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Modular Design, Generalized Inverses, and Convex Programming

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Modular Design, Generalized Inverses, and Convex Programming

by
Abraham Charnes
Northwestern University, Evanston, Ill
and
Michael J. L. Kirby
Research Analysis Corporation

RESEARCH ANALYSIS CORPORATION
McLEAN, VIRGINIA

FOREWORD

This paper presents an application of a generalized inverse to a particular mathematical programming problem, the modular design problem. It is designed to illustrate the simplification that can often result by transforming a given problem into a more convenient form. In particular the use of a generalized inverse in making such a transformation is emphasized.

The results of this paper are closely related to the work of A. V. Fiacco and G. P. McCormick of RAC on nonlinear programming. In addition the method of separable convex programming under linear equality constraints developed by J. E. Falk of RAC is used in the solution of the transformed problem.

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Nicholas M. Smith
Chief, Advanced Research Division

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Modular Design, Generalized Inverses, and Convex Programming

ABSTRACT

It is shown that the modular design problem
minimize

$$\sum_{i=1}^{m} y_i \quad \sum_{j=1}^{n} z_j$$

subject to

$$y_i z_j \geq r_{ij}$$

$$y_i, z_j \geq 0 \quad i = 1, \ldots, m, \; j = 1, \ldots, n.$$

can be transformed into a problem of minimizing a separable convex function subject to linear equality constraints and nonnegativities. This transformation is effected by using a generalized inverse of the constraint matrix. Moreover the nature of the functional and the constraints of the separable problem are such that a good starting point for its solution can be obtained by solving a particular transportation problem. Several possible methods for solving the separable problem are discussed, and the results of our computational experience with these methods are given. It is also shown that the modular design problem can be viewed as a special case of a large class of general engineering design problems that have been discussed in the literature.

INTRODUCTION

The problem to be considered is the following:
minimize

$$\left(\sum_{i=1}^{m} e_i E_i\right) \left(\sum_{j=1}^{n} d_j D_j\right)$$

subject to

$$\begin{aligned} & E_i D_j \geq R_{ij} \quad i = 1, \ldots, m \\ & \qquad\qquad\quad j = 1, \ldots, n \\ & E_i, D_j \geq 0. \end{aligned} \tag{0}$$

In Eq 0 it is assumed that e_i, d_j, R_{ij} are constants such that $e_i > 0$, $d_j > 0$, $R_{ij} \geq 0$, for all i, and j and E_i, D_j are the variables.† Using the transformations $y_i \equiv e_i E_i$ $i = 1, \ldots, m$, $z_j \equiv d_j D_j$, $j = 1, \ldots, n$ and $r_{ij} \equiv e_i d_j R_{ij}$ $i = 1, \ldots, m$ $j = 1, \ldots, n$ the problem can be written in the form:
minimize

$$\sum_{i=1}^{m} y_i \sum_{j=1}^{n} z_j$$

subject to

$$\begin{aligned} & y_i z_j \geq r_{ij} \quad i = 1, \ldots, m \\ & \qquad\qquad\; j = 1, \ldots, n \\ & y_i, z_j \geq 0, \end{aligned} \tag{1}$$

where r_{ij} are constants such that $r_{ij} \geq 0$ for all i and j. In Eq 1 it can be assumed that y_i, z_j are strictly positive without loss of generality since for one of these to be zero it is necessary for a row or column of the (r_{ij}) matrix to be zero, and if this were so that row or column and its corresponding variable could be dropped from the problem.

Equation 0, or equivalently Eq 1, may be called the <u>modular design problem</u>. This problem was first suggested by Evans.[1] In his paper Evans presented an algorithm for solving the problem that "converges rather slowly" to the optimal solution. He also comments that the algorithm "can be made to converge much faster if one makes intelligent guesses in the trial solution phase; however there is apparently no way to program this last."[1]

This paper shows that the constraints y_i, $z_j > 0$, where strict inequality exists, permit Eq 1 to be transformed into a problem of minimizing a convex function subject to linear inequalities. Then a generalized inverse is used to effect a second transformation, which yields a problem in the form of minimizing a <u>separable</u> convex function subject to linear equality constraints in nonnegative variables.

The achievement of a problem involving a separable convex functional means that direct contact is established with the computing routines of linear

†For an illustration of a situation where such a problem arises the reader is referred to Ref 1.

programming by recourse to suitable piecewise linear approximations. A discussion of such a technique is given in Chap. X of Charnes and Cooper's Management Models and Industrial Applications of Linear Programming[2] and in various references cited therein.

Moreover the nature of the functional and the constraints of the separable problem are such that a good starting point for its solution can be obtained by solving a particular distribution problem. Since there are extremely efficient algorithms for solving such problems an initial solution can be quickly obtained, thus eliminating one of the major difficulties discussed by Evans.[1]

Finally it is shown that Eq 0 is a member of a large class of engineering design problems that have been considered in several research papers produced at Northwestern University and Westinghouse Research Laboratories.[3–6]

MATRIX THEORY

Definition. Let A be any $m \times n$ matrix.

Then any $n \times m$ matrix $A^{\#}$ such that $AA^{\#}A = A$ will be called a generalized inverse of A.

A generalized inverse is unique if and only if A is square and nonsingular. It can also be shown that $A^{\#}$ is a generalized inverse of A if and only if $X = A^{\#}b$ is a solution of $AX = b$, whenever $AX = b$ is consistent. Further properties of $A^{\#}$ can be found in papers by Rao and by Charnes and Kirby.[7,8]

Lemma 1. A necessary and sufficient condition that the matrix equation $AX = b$ be consistent is that $AA^{\#}b = b$.

Proof. Suppose that $AA^{\#}b = b$. Then $X = A^{\#}b$ is a solution of $AX = b$, implying the consistency of $AX = b$.

Conversely suppose that $AX = b$ is consistent. Then

$$b = AX \Rightarrow AA^{\#}b = AA^{\#}AX = AX = b$$

Lemma 2. Let $\Psi \equiv \{X \mid AX \geq c\}$.

Let $\Omega \equiv \{\omega \mid \omega \geq 0,\ AA^{\#}(c+\omega) = c+\omega\}$.

Then $X \in \Psi \Leftrightarrow$ there exists an ω such that $AX = c + \omega$ and $\omega \in \Omega$.

Proof. Let $X \in \Psi$.

Let $\omega \equiv AX - c$, so that, for any X, ω is uniquely defined.

Then $\omega \geq 0$ as $X \in \Psi \Leftrightarrow AX \geq c$. Also by this definition of ω $AX = c + \omega$. But by lemma 1, $AX = c + \omega \Leftrightarrow AA^{\#}(c+\omega) = c+\omega$. So that for any $X \in \Psi$ the ω defined by $\omega = AX - c$ satisfies $\omega \in \Omega$ and $AX = c + \omega$.

Conversely for any $\omega \in \Omega$, $AA^{\#}(c+\omega) = c + \omega$, which by lemma 1 implies there exists X such that $AX = c + \omega$ is consistent. Also $\omega \in \Omega \Leftrightarrow \omega \geq 0$, hence $AX = c + \omega \Leftrightarrow AX \geq c$. Thus for any $\omega \in \Omega$, there exists an X (which is not necessarily unique) such that $AX = c + \omega$ and $X \in \Psi$.

Therefore if $AX \geq c$ are the constraints of some mathematical programming problem, this problem can be transformed into an equivalent one whose constraints are $AX = c + \omega$, $AA^{\#}(c+\omega) = c + \omega$, and $\omega \geq 0$, providing the following pair of equivalent programming problems.

optimize

$$f(X)$$

subject to

$$AX \geq c$$

and optimize

$$f(X)$$

subject to

$$AX = c + \omega$$

$$AA^{\#}(c + \omega) = c + \omega$$

$$\omega \geq 0.$$

The added constraints $AA^{\#}(c + \omega) = c + \omega$, $\omega \geq 0$ in this second problem ensure that only vectors ω of "slack variables" for which the system $AX = c + \omega$ is consistent will be considered.

The reason why such consistency conditions are not mentioned in most linear programming texts is that A is assumed to be an $m \times n$ matrix of rank m. Since it can be shown that $AA^{\#} = I$, the $m \times m$ identity matrix, if and only if A has full row rank,† it can be seen that when A has rank m the constraints $AA^{\#}(c + \omega) = c + \omega$ become $c + \omega = c + \omega$; thus they are redundant and can be ignored.

MODULAR DESIGN PROBLEM

In the constraints of Eq 1, $y_i > 0$, $z_j > 0$ $i = 1, \ldots, m$, $j = 1, \ldots, n$. Since these are strict inequalities the following change of variables can be made:

$$\text{Let } y_i = e^{u_i} \quad i = 1, \ldots, m$$

$$\text{Let } z_j = e^{v_j} \quad j = 1, \ldots, n$$

where u_i, v_j are unrestricted variables. Then Eq 1 becomes minimize

$$\sum_{i=1}^{m} e^{u_i} \sum_{j=1}^{n} e^{v_j}$$

subject to

$$e^{u_i} e^{v_j} \geq r_{ij} \quad \begin{array}{l} i = 1, \ldots, m \\ j = 1, \ldots, n \end{array}$$

This is equivalent to
minimize

$$\sum_{i=1}^{m} \sum_{j=1}^{n} e^{u_i + v_j}$$

subject to

$$u_i + v_j \geq c_{ij} \quad \begin{array}{l} i = 1, \ldots, m \\ j = 1, \ldots, n \end{array} \tag{2}$$

where $c_{ij} \equiv \ln(r_{ij})$ $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Since it is possible that some $r_{ij} = 0$, the value of $-\infty$ is an admissible value for c_{ij}. Whenever $c_{ij} = -\infty$, the constraint $u_i + v_j \geq c_{ij}$ is redundant

†For a proof see Ref 7, lemma 1.

corresponding to any constraint of Eq 1 of the form $y_i z_j \geq 0$ being redundant as $y_i, z_j > 0$. A procedure for avoiding the computational difficulties that arise in dealing with the quantity $-\infty$ will be given later.

The constraint set of Eq 2 can be written in matrix form as $AX \geq c$, where

$$c^T = (c_{11}, \ldots, c_{1n}, \ldots, c_{i1}, \ldots, c_{ij}, \ldots, c_{in}, \ldots, c_{m1}, \ldots, c_{mn})$$
$$X^T = (u_1, \ldots, u_i, \ldots, u_m, v_1, \ldots, v_j, \ldots, v_n)$$

and the $mn \times (m+n)$ matrix $A \equiv (a_{rs})$, $r = 1, \ldots, mn$, $s = 1, \ldots, m+n$ has the following properties:

$$\text{(i)} \quad a_{kn+j,\, k+1} = +1 \quad j = 1, \ldots, n; \quad k = 0, \ldots, m-1$$

$$\text{(ii)} \quad a_{kn+j,\, m+j} = +1 \quad j = 1, \ldots, n; \quad k = 0, \ldots, m-1$$

and all remaining a_{rs} are zero.†,[1]

Let ω be the $mn \times 1$ vector defined by

$$\omega^T = (\omega_{11}, \ldots, \omega_{1n}, \ldots, \omega_{i1}, \ldots, \omega_{ij}, \ldots, \omega_{in}, \ldots, \omega_{m1}, \ldots, \omega_{mn}).$$

Then using the results of the previous section Eq 2 can be replaced by the following equivalent problem:

minimize

$$\sum_{i=1}^{m} \sum_{j=1}^{m} e^{u_i + v_j}$$

subject to

$$\begin{aligned} AX &= c + \omega \\ AA^{\#}(c+\omega) &= c + \omega \\ \omega &\geq 0. \end{aligned} \tag{3}$$

From definitions of A, X, c, and ω it can be seen that $AX = c + \omega$ can be written as $u_i + v_j = c_{ij} + \omega_{ij}$ for all i, j. Thus the u_i, $i = 1, \ldots, m$, v_j, $j = 1, \ldots, n$ can be eliminated from Eq 3 and rewritten as

minimize

$$\sum_{i=1}^{m} \sum_{j=1}^{m} e^{c_{ij} + \omega_{ij}}$$

subject to

$$\begin{aligned} AA^{\#}(c+\omega) &= c + \omega \\ \omega &\geq 0. \end{aligned} \tag{4}$$

where the constraints of Eq 4 guarantee that u_i, v_j exist that satisfy $u_i + v_j = c_{ij} + \omega_{ij}$ and $u_i + v_j \geq c_{ij}$.

In other words Eq 4 can be solved for ω_{ij}^*, the optimal values of ω_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, and then the system of linear equations $u_i + v_j = c_{ij} + \omega_{ij}^*$ $i = 1, \ldots, m$, $j = 1, \ldots, n$ can be solved to obtain u_i^*, v_j^* the optimal values of u_i, $i = 1, \ldots, m$ and v_j, $j = 1, \ldots, n$.

Moreover, because A has a specific form one of its generalized inverses can be characterized in the following manner: Let $A^{\#} \equiv (a^{\#}_{s,r})$, $s = 1, \ldots, m+n$, $r = 1, \ldots, mn$; then $A^{\#}$ is such that

†In this characterization of A it is assumed that $m, n \geq 2$. If $m = 1$, or $n = 1$, or both, the problem is trivial.

(i) $a_{l,1} = -1$ $\quad l = 2, \ldots, m$

(ii) $a^{\#}_{m+j,j} = 1$ $\quad j = 1, \ldots, n$

(iii) $a^{\#}_{k+1,kn+1} = 1$ $\quad k = 1, \ldots, m-1$

and all remaining $a^{\#}_{s,r}$, $s = 1, \ldots, m+n$, $r = 1, \ldots, mn$ will be zero. It can be shown that $A^{\#}$ defined above does indeed have the property that $AA^{\#}A = A$.

Using this $A^{\#}$ and the study's characterization of A, $AA^{\#}$ can be computed and finally $(I - AA^{\#})$, where I is the $mn \times mn$ identity matrix. If T is defined $\equiv$ $(I - AA^{\#})$ then the $mn \times mn$ matrix $T \equiv (t_{p,q})$, $p = 1, \ldots, mn$, $q = 1, \ldots, mn$ has the following properties:

(i) $t_{kn+l,1} = 1$ $\quad k = 1, \ldots, m-1, l = 2, \ldots, n$

(ii) $t_{kn+l,l} = -1$ $\quad k = 1, \ldots, m-1, l = 2, \ldots, n$

(iii) $t_{kn+l,kn+1} = -1$ $\quad k = 1, \ldots, m-1, l = 2, \ldots, n$

(iv) $t_{kn+l,kn+l} = 1$ $\quad k = 1, \ldots, m-1, l = 2, \ldots, n$

and all remaining $t_{p,q}$, $p, q, = 1, \ldots, mn$ will be zero.

Finally replacing the constraints $AA^{\#}(c + \omega) = c + \omega$ of Eq 4 by $(I - AA^{\#})$ $\omega = -(I - AA^{\#})c$ and using the study's characterization of T Eq 4 can be written as

minimize

$$\sum_{i=1}^{m} \sum_{j=1}^{n} e^{c_{ij} + \omega_{ij}} = \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ij} e^{\omega_{ij}}$$

subject to

$$\omega_{11} - \omega_{1,l} - \omega_{k+1,1} + \omega_{k+1,l} = -c_{11} + c_{1,l} + c_{k+1,1} - c_{k+1,l}$$

$$k = 1, \ldots, m-1, \; l = 2, \ldots, n \qquad (5)$$

$$\omega_{ij} \geq 0 \quad i = 1, \ldots, m \quad j = 1, \ldots, n$$

The first objective, which was to write Eq 0 in the form of minimizing a separable convex function subject to linear equality constraints in nonnegative variables, has now been accomplished.

DUAL DISTRIBUTION APPROXIMATION

Prior to discussing various ways of solving Eq 5, how to obtain an initial feasible solution quickly will be shown. Since in Eq 2 it is required that $u_i + v_j \geq c_{ij}$, and at the same time it is desired to minimize $\Sigma_{i=1}^{m} \Sigma_{j=1}^{n} e^{u_i + v_j}$ it is expected that the optimal solution will have $u_i + v_j \approx c_{ij}$. Thus the initial feasible solution will be obtained by using the approximation

$$e^{u_i + v_j} = e^{c_{ij}} \cdot e^{u_i + v_j - c_{ij}}$$

$$\approx e^{c_{ij}} (1 + u_1 + v_j - c_{ij}).$$

In other words $e^{u_i + v_j - c_{ij}}$ is going to be approximated by the linear part of its Taylor series expansion about zero. Thus a feasible solution to Eq 2 is obtained by replacing $\Sigma_{i=1}^{m} \Sigma_{j=1}^{n} e^{u_i + v_j}$ by

$$\sum_{i=1}^{m} \sum_{j=1}^{n} e^{c_{ij}}(1 + u_i + v_j - c_{ij})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (1 - c_{ij}) e^{c_{ij}} + \sum_{i=1}^{m} u_i \left(\sum_{j=1}^{n} e^{c_{ij}} \right) + \sum_{j=1}^{n} v_j \left(\sum_{i=1}^{m} e^{c_{ij}} \right)$$

and then solving the problem
minimize

$$\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

subject to

$$u_i + v_j \geq c_{ij} \tag{6}$$

where

$$a_i = \sum_{j=1}^{n} e^{c_{ij}}, \; b_j = \sum_{i=1}^{m} e^{c_{ij}} \quad \text{and} \quad \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j .$$

$$= \sum_{j=1}^{n} r_{ij} \qquad = \sum_{i=1}^{m} r_{ij}$$

This is the dual of the distribution problem
maximize

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j = 1, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = a_i \qquad i = 1, \ldots, m$$

$$x_{ij} \geq 0 \qquad i = 1, \ldots, m, \; j = 1, \ldots, n. \tag{7}$$

Since extremely efficient algorithms exist for solving distribution problems, Eq 7, and hence Eq 6, can be solved quickly. If the optimal values of u_i, v_j for Eq 6 are denoted by u_i^o, v_j^o, the relation $\omega_{ij}^o = u_{ij}^o + v_j^o - c_{ij}$ can then be used to define the initial values of ω_{ij} for Eq 5.

The starting point that is obtained by this procedure will, in general, give values of ω_{ij}^o that are very close to the optimal values of ω_{ij} for some ij. In addition if a solution exists of the system $u_i + v_j = c_{ij}$ $i = 1, \ldots, m$ $j = 1, \ldots, n$, (i.e., if $\omega = 0$ is feasible for Eq 5), this point will be optimal for both Eqs 6 and 5; hence the initial feasible solution will be optimal.

In any case, because of the convexity of the objective function of Eq 5 in ω_{ij}, the iteration of this Taylor expansion procedure about ω_{ij}^o to get a new dual distribution approximation, and so on, appears to be a possibly worthwhile method in its own right for the solution of the modular design problem.

UNIQUENESS OF SOLUTIONS

As Evans[1] has pointed out, the solution of Eq 1 is not unique. If y_i^*, z_j^* is a minimizing solution for Eq 1, then so is θy_i^*, z_j^*/θ for any $\theta > 0$. This non-

uniqueness can be seen again in Fq 2 since if u_i^*, v_j^*, optimize Eq 2 then so does $u_i^* + \gamma$, $v_j^* - \gamma$ for any constant γ.

This is not true however for problem 5, for the optimal solution of Eq 5 will be unique if the function $\Sigma_{i=1}^{m} \Sigma_{j=1}^{n} e^{c_{ij} + \omega_{ij}}$ is a strictly convex function of ω_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$. A necessary and sufficient condition for this to be true is that all $c_{ij} > -\infty$, i.e., that $r_{ij} > 0$ or $(R_{ij} > 0)$ for all i, j. Since we must have $y_i, z_j > 0$, it follows that for any pair (i, j) for which $r_{ij} = 0$ the constraint $y_i z_j \geq r_{ij}$ will be redundant. Thus one way of assuring that the functional of Eq 5 is strictly convex is to replace $r_{ij} = 0$ by $\tilde{r}_{ij} = \epsilon$ where $\epsilon > 0$ is very small. The solution of this perturbed problem will coincide with the solution of Eq 5 if ϵ is very small in con.parison with the nonzero r_{ij}.†

The advantage of strict convexity in the functional of Eq 5 is that it permits the use of various nonlinear programming routines that require this strictly convex property. Examples of such algorithms appear in papers by Falk, Fiacco and McCormick, Rosen, and Charnes and Lemke.[9–13]

RELATION TO THE GENERAL ENGINEERING DESIGN PROBLEM

The transformation from Eq 1 to Eq 5 shows that the modular design problem is simply a special case of a large class of engineering design problems that have been discussed elsewhere in the literature. Charnes and Cooper[3,4] show that many problems in determining the optimal parameters for an engineering design can be expressed mathematically as a problem of minimizing a separable convex functional subject to linear constraints. This is exactly the form in which Eq 5 is given.

Moreover Charnes and Cooper[3] discuss various computational approaches to such problems other than the methods mentioned above. In particular they suggest that one means of solving such problems would be to get into the neighborhood of the optimum as soon as possible and then undertake further refinements after attaining this neighborhood. This approach would seem to be highly feasible here because of the ease with which Eq 6 can be solved.

Equation 0 is also directly related to the work on engineering design problems done by Duffin, Peterson, and Zener.[5,6] In their papers they consider problems of the form

minimize

$$\sum_{i=1}^{n_0} c_i t_1^{a_{i1}} \cdot t_2^{a_{i2}} \cdots t_m^{a_{im}}$$

subject to

$$\begin{aligned} &\sum_{i=m_k}^{n_k} c_i t_1^{a_{i1}} \cdots t_m^{a_{im}} \leq 1 \quad k = 1, \ldots, p \\ &t_j > 0 \qquad\qquad j = 1, \ldots, m. \end{aligned} \tag{8}$$

In Eq 8 it is assumed that $c_i > 0$ all i, $m_1 = n_0 + 1$, $m_2 = n_1 + 1$, ... etc. and a_{ij} are any real constants.

†A sufficient condition for the solution of the perturbed problem to be optimal for Eq 1 is that the constraint $y_i z_j \geq \tilde{r}_{ij} = \epsilon$ not be binding, i.e., $y_i^* z_j^* > \tilde{r}_{ij} = \epsilon$.

Equation 1 can be written in this form:
minimize

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{(i-1)n+j}\, t_i\, t_{m+j}$$

subject to

$$\begin{aligned} \tilde{c}_{mn+(i-1)n+j}\, t_i^{-1}\, t_{m+j}^{-1} &\le 1 \\ t_i > 0\; t_{m+j} &> 0, \end{aligned} \tag{9}$$

where

$$t_i \equiv y_i \; i=1, \ldots, m, \; t_{m+j} \equiv z_j \; j=1, \ldots, n, \; \tilde{c}_{(i-1)n+j} \equiv 1$$

and

$$\tilde{c}_{mn+(i-1)n+j} \equiv r_{ij} \; i=1, \ldots, m, \; j=1, \ldots, n.$$

Duffin, Peterson,and Zener then write a dual to Eq 8 and proceed to show that in many cases the solution of the dual is much easier to obtain than that of the primal, Eq 8. Using their duality theory and a transformation of the dual functional employed by Charnes and Cooper,[3] which yields a concave functional, the dual of Eq 9 is
maximize

$$\sum_{l=1}^{mn} \delta_l\, [\ln(\tilde{c}_l) - \ln(\delta_l)] + \sum_{l=mn+1}^{2mn} \delta_l \ln(\tilde{c}_l)$$

subject to

$$\sum_{l=1}^{mn} \delta_l = 1$$

$$\sum_{l=(j-1)n+1}^{jn} \delta_l - \sum_{l=(m+j-1)n+1}^{(m+j)n} \delta_l = 0 \qquad j=1, \ldots, m$$

$$\sum_{k=1}^{m} \delta_{(k-1)n-m+i} - \sum_{k=1}^{m} \delta_{(m+k-1)n-m+i} = 0 \qquad i=m+1, \ldots, m+n$$

$$\delta_l > 0 \qquad l=1, \ldots, mn$$

$$\delta_l, \ge 0 \qquad l=mn+1, \ldots, 2mn. \tag{10}$$

Since the constraints of Eq 10 involve $m+n+1$ equalities and there are $2\,mn$ variables, there will be only one feasible point for Eq 10 if $2\,mn = m+n+1$. This will be true only when $m=1$, $n=2$ (or $n=1$, $m=2$), in which case the solution of Eq 0 is trivial anyway. Thus it would appear that although the dual to Eq 8 is very useful for some engineering design problems it does not simplify the solution of the modular design problem.

However, this method does have the useful property that any feasible point for Eq 10 yields a value for its functional that is a lower bound on the functional of Eq 0.[6] Since any feasible point for Eq 0 (in particular that point obtained by solving Eq 6) yields an upper bound on the functional, it can be seen that one might quickly get bounds on the optimal value of Eq 0 by solving Eq 6 and getting any feasible solution to Eq 10.

EXAMPLE

As an example of the modular design problem consider the following, which appeared in the Evans paper.[1] The elements in Table 1 are the r_{ij} values for $i = 1, 2, 3, 4$; $j = 1, 2, 3$. Then the corresponding set of c_{ij} values is obtained using the relation $c_{ij} \equiv \ln(r_{ij})$. In accordance with the procedure suggested above $r_{23} = 0$ is replaced by $\bar{r}_{23} = \epsilon = 1$. This problem is then solved and as long as at the optimum $u_2^* + v_3^* > 1$, the optimal solution to the problem is $r_{23} = 0$.

The c_{ij} $i = 1, \ldots, 4$ $j = 1, 2, 3$, are shown in Table 2.

TABLE 1

$r_{11} = 15$	$r_{12} = 23$	$r_{13} = 44$
$r_{21} = 13$	$r_{22} = 13$	$r_{23} = 0$ ($\bar{r}_{23} = 1$)
$r_{31} = 15$	$r_{32} = 17$	$r_{33} = 35$
$r_{41} = 34$	$r_{42} = 12$	$r_{43} = 22$

TABLE 2

$c_{11} = 2.70805$	$c_{12} = 3.13549$	$c_{13} = 3.78419$
$c_{21} = 2.56495$	$c_{22} = 2.56495$	$c_{23} = 0.00000$
$c_{31} = 2.70805$	$c_{32} = 2.83321$	$c_{33} = 3.55535$
$c_{41} = 3.52636$	$c_{42} = 2.48491$	$c_{43} = 3.09104$

Next the solution of Eq 6 is obtained. To do this the problem is set up in tabular form as shown in Table 3. The optimal solution of Eq 6 is obtained in one pass through the algorithm given in (Ref 2, pp 57–63).

TABLE 3

	$v_1^o = 2.56495$	$v_2^o = 2.56495$	$v_3^o = 3.21365$	
$u_1^o = 0.57054$	2.70805	3.13549 (47)	3.78419 (35)	$a_1 = 82$
$u_2^o = 0.00000$	2.56495 (9)	2.56495 (18)	0.00000	$a_2 = 27$
$u_3^o = 0.34170$	2.70805	2.83321	3.55535 (67)	$a_3 = 67$
$u_4^o = 0.96141$	3.52636 (68)	2.48491	3.09104	$a_4 = 68$
	$b_1 = 77$	$b_2 = 65$	$b_3 = 102$	

In Table 3 the numbers in the upper left-hand corner of each box represent the c_{ij}. The numbers in circles are the optimal values of x_{ij} in Eq 7; where no circle appears it means that the corresponding optimal x_{ij} is zero. One set of optimal u_i, v_j are shown to the left and across the top of the table. The corresponding set of $\omega_{ij}^o = u_i^o + v_j^o - c_{ij}$ are given in Table 4.

Finally, using the starting point given in Table 4 the problem is solved using the method developed in Falk's "An Algorithm for Separable Convex Programming under Linear Equality Constraints."[9] The optimal ω_{ij}, u_i, v_j and corresponding optimal values of $y_i z_j$ are shown in Table 5.

It can be seen from the table that $\omega^*_{23} > 0$ so that the constraint $u_2 + v_3 \geq \bar{r}_{23}$ is not binding; hence the above solution is optimal for the problem where $r_{23} = 0$.

TABLE 4

$\omega^o_{11} = 0.42744$	$\omega^o_{12} = 0.00000$	$\omega^o_{13} = 0.00000$
$\omega^o_{21} = 0.00000$	$\omega^o_{22} = 0.00000$	$\omega^o_{23} = 3.21365$
$\omega^o_{31} = 0.19860$	$\omega^o_{32} = 0.07344$	$\omega^o_{33} = 0.00000$
$\omega^o_{41} = 0.00000$	$\omega^o_{42} = 1.04145$	$\omega^o_{43} = 1.06402$

TABLE 5

	$v^*_1 = 1.84635$	$v^*_2 = 1.54578$	$v^*_3 = 2.19448$
$u^*_1 = 1.58971$	$\omega^*_{11} = 0.72801$ $y^*_1 z^*_1 = 31.06$	$\omega^*_{12} = 0.00000$ $y^*_1 z^*_2 = 23.00$	$\omega^*_{13} = 0.00000$ $y^*_1 z^*_3 = 24.87$
$u^*_2 = 1.01917$	$\omega^*_{21} = 0.30057$ $y^*_2 z^*_1 = 17.56$	$\omega^*_{22} = 0.00000$ $y^*_2 z^*_2 = 13.00$	$\omega^*_{23} = 3.21365$ $y^*_2 z^*_3 = 24.87$
$u^*_3 = 1.36087$	$\omega^*_{31} = 0.49917$ $y^*_3 z^*_1 = 24.71$	$\omega^*_{32} = 0.07344$ $y^*_3 z^*_2 = 18.29$	$\omega^*_{33} = 0.00000$ $y^*_3 z^*_3 = 35.00$
$u^*_4 = 1.68001$	$\omega^*_{41} = 0.00000$ $y^*_4 z^*_1 = 34.00$	$\omega^*_{42} = 0.74038$ $y^*_4 z^*_2 = 25.17$	$\omega^*_{43} = 0.78345$ $y^*_4 z^*_3 = 48.16$

Note that the initial starting point given in Table 5 actually gives seven of the variables at their optimum values. Thus the solution of the dual distribution approximation does indeed give a very good starting point for the solution of Eqs 2 or 5.†

As suggested in the section "Dual Distribution Approximation" the dual distribution approximation was solved a second time by approximating $u_i + v_j - \omega^o_{ij} - c_{ij} \equiv u_i + u_j - u^o_i - u^o_j$ by the linear part of its Taylor series about zero. The result was an improved starting point (i.e., it gave a lower value of the objective function for Eq 5 than did ω^o_{ij}) and six of the resulting ω_{ij} were at their optimum values.

†See Falk's paper[9] for further examples and some results of computational experience on problems similar to Eq 5.

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